APPOXIMATIONS FOR ANY TRIGONOMETRIC FUNCTION USING LEGENDRE POLYNOMIALS

About a decade ago(see https://www2.mae.ufl.edu/~uhk/TRIG-APPROX-PAPER.pdf) while studying definite integrals involving the product of rapidly oscillating Legendre Polynomials and a second slowly varying function f(ax) in the range 0<x<1, we came up with the following important integral and its solution-

$$J(n,a)=\int_{x=0}^{1} P(2n+1,x)\sin(ax)\,dx = [\frac{1}{a^{2n+2}}][N(n,a)sin(a) - M(n,a)\cos(a)]$$

Here N and M are polynomials in n and a, while P(2n+1,x) are the odd Legendre Polynomials which have value of P(2n+1,0)=0 and P(2n+1,1)=1 with a total of n zeroes in 0<a<1. If one now lets n become large, the value of J(n,a) , when multiplied by the power of a^(2n+2) , approach zero, leaving one with the approximations-

$$\tan(a) \approx T(n,a) = M(n,a)/N(n,a)$$

The values for explicit values for the polynomials M(n,a) and N(n,a)] are easiest to find using the operation-

collect[J(n,a),{sin(a),cos(a)}] .

The larger n is taken the longer the quotient for T(n,a) becomes. The above method is now referred to in the literature as the KTL Method (for Kurzweg, Timmins, Legendre) . Once that approximation for tan(a) is found, other functions such as sin(a) and cos(a) follow directly from the identities-

$$\sin(a) \approx S(n,a) = \frac{M(n,a)}{\sqrt{M(n,a)^2+N(n,a)^2}} \qquad and \qquad \cos(a) \approx C(n,a)\frac{N(n,a)}{\sqrt{M(n,a)^2+N(n,a)\text{^}2}}$$

We have evaluated T(n,a) for n=1,2,3, 4, and 5. Here are the results-

N(1,a)= 6a^2-15

M(1,a)= a^3-15a

N(2,a)= 15a^4-420a^2+945

M(2,a)= a^5-105a^3+945

N(3,a) = 28a^6-3150a^4+62370a^2-135135

M(3,a)= a^7-378a^5+17325a^3-135135a

N(4,a)= 45a^8-13860a^6+945945a^4-16216200a^2+34459425

M(4,a)= a^9-990a^7+135135a^5-4729725a^3+34459425a

N(5,a)= 66a^10-45045a^8+7567560a^6-413513100a^4+6547290750a^2-13749310575

M(5,a)=a^11-2145a^9+675675a^7-64324260a^5+1964187225a^3-13749310575a

One sees that the polynomials for N(n,a) start with a^(2n) and those for M(n,a) with a^(2n+1). This implies rapid polynomial length increase with n. The accuracy of the approximations also increase with increasing length. A measure of the technique's accuracy for T(n,a)=M(n,a)/N(n,a) is indicated by looking at the simple a=1 case. Here are the tan(1) estimates-

n	T(n,1)
1	1.55
2	1.557407
3	1.55740772
4	1.557407724654902
5	1.557407724654902230506

These results should be compared with the exact value-

tan(1)= 1.5574077246549022305069748074583…

It shows that the approximations for the tangent of one radian to be good to 15 places after the decimal for n=4 and to 21 places for n=5. We have cut-off the approximations in the above T(n,1) table at the point they first depart from tan(1).

An important property of tan(a) is that-

tan(π/4-Δ)tan(π/4+Δ)=1

Thus ,to evaluate tan(a) at any point, it will be sufficient to look at only the limited range 0<a<π/4=0.785398 keeping Δ at less than π/4. Evaluation of trigonometric functions in this range is well suited for the KTL Method.

Next we get estimates for the other two most important trigonometric functions sin(a)≈S(n,a) and cos(a)≈C(n,a). We confine ourselves to n=4 and 'a'=1. Here we expect an accuracy of about 15 decimal places. Substituting in the values of N(4,1) and M(4,1), we find-

$$\sin(1) \approx \frac{M(4,1)}{\sqrt{N(4,1)^2+M(4,1)^2}} = 0.841470984807896$$

and

$$\cos(1) \approx \frac{N(4,1)}{\sqrt{N(4,1)^2+M(4,1)\text{^}2)}} = 0.5403023058681397$$

, where we have retained only the first 15 and 16 digits agreeing with the exact values. The accuracy of our S(n,a) and C(n,a) approximations will improve considerably as 'a' is lowered from a=1. So, for example, sin(π/6)=sin(30deg) will yield the n=4 result-

S(4,π/6)=0.4999999999999999999997

This is accurate to 21 places when compared to the exact solution sin(π/6)=0.5.

We have shown above that the KTL Method works great for values of ‘a’ lying in the range $0<a<1$ and that the n=4 approximations will bring any trigonometric combination of tan(a), sin(a), and cos(a) in this ‘a’ range to better than 15 decimal place accuracy. These results are more accurate than any extant trigonometric math tables.

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