ISAAC NEWTON AND THE BINOMIAL FORMULA

Around 1676 the famous mathematician and physicist Isaac Newton first played around with the known Binomial Theorem-

$$(1+x)^n=1+nx+n(n-1)x^2/2!+n(n-1)(n-2)x^3/3!+\ldots=\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}x^m$$

which had been examined extensively earlier for the case where n is a positive integer. Under those conditions the formula yields nth order polynomials whose coefficients are equivalent to elements in a standard Pascal Triangle. Thus –

$$(1+x)^4=1+4x+6x^2+4x^3+x^4$$

What Newton realized is that x can be replaced by f(x) and n take on negative and non-integer values. This observation allowed him to come up many additional expansions. We want in this article to re-derive some of these.

Lets start with x=-x and n=(-1). This produces-

$$1/(1-x)=1+x+x^2+x^3+x^4+O(x^5)$$

It represents the geometric series which converges for 0<x<1. Replacing x by exp(-x),yields the additional form-

$$1/[1-\exp(x)]=\sum_{m=0}^{\infty}\exp(-mx)$$

Next take x=1 and n=1/2. Here we have-

$$\mathrm{sqrt}(2)=1+1/2-1/8+1/16-5/128+7/256-\ldots$$

To get an estimate for π we look at the area of a pie sliced area with small angle π/6 of a unit radius circle . It reads, after some transformations,-

$$\pi/12=\mathrm{sqrt}(3)/8+\left(\frac{1}{2}\right)\int_{u=0}^{1/4} sqrt(u/(1-u))\,d\,u$$

Next expanding the sqrt term in the integral produces-

$$\mathrm{sqrt}[u/(1-u)]=u^{(1/2)}+u^{(3/2)}/2+3u^{(5/2)}/8+5u^{(7/2)}/16$$

Upon integrating and applying the limits we have-

$$\pi=3\mathrm{sqrt}(3)/2+1/2+3/80+9/1792+..=3.14059$$

The series offers a lower bound on π=3.1415926… but is not as good as certain arctan formulas, AGM methods , or iteration techniques for quickly finding accurate values for the irrational constant π to a large number of decimal places.

Another use Newton found for the Binomial Formula is the approximation for certain integrals. Take for example the integral-

$$J=\int_{t=0}^{1}\frac{dx}{1+x^2}=\arctan(1)=\pi/4$$

On expanding 1/(1+x^2) as a Binomial Series and then integrating and putting in the limits we find-

J=1-1/3+1/5-1/7+1/9-

So taking this expansion out to infinity one finds the famous Gregory Formula-

$$\pi/4=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=0.785398\ldots$$

This is the simplest expression for π known. Unfortunately it is also one of the slowest converging series .

Consider next the integral-

$$K=\int_{x=0}^{1}1/\sqrt{1+x^2}\,dx=\sinh^{-1}(1)=0.88137358\ldots$$

Expanding the radical as a Binomial Series we find-

$$K=\int_{x=0}^{1}\frac{dx}{sqrt(1+x^2)}=\int_{x=0}^{1}\{1-x^2/2+3x^4/8-5x^6/16+35x^8/128\}dx=1-1/6+3/40-5/112+35/1152-\ldots$$

On adding together the first five terms of the series , we find K=0.8940724206.

As a final application of Newton's use of the Binomial Theorem consider the following integral and its numerical solution-

$$L=\int_{x=0}^{1}\sqrt{\frac{(1-x^2)}{(1+x^2)}}\,dx=0.711958659\ldots$$

To get an analytical approximation we first expand the radical as-

1-x^2+x^4/2-x^6/2+3x^8/8-3x^10/8+5x^12/16-…

Then, upon integrating and using the indicated limits, we find-

L=1-1/3+1/10-1/14+3/72-3/88+5/208=0.717236

This result is accurate to just two decimal places. It again emphasizes the slowness of the convergence of a typical Binomial Expansion.

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