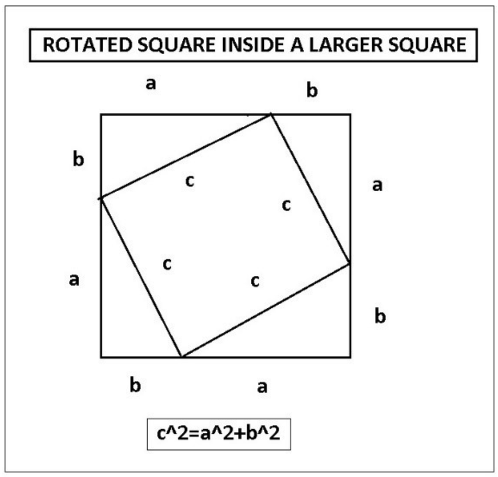
LAW OF COSINES AND SINES

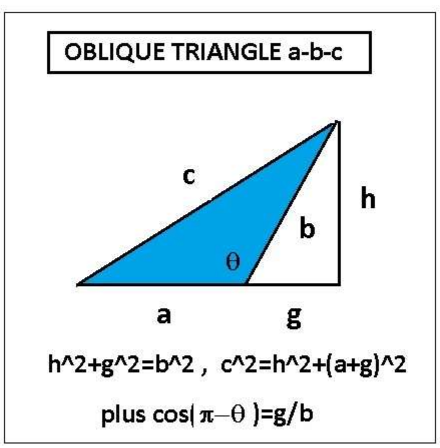
Two of the most important laws of trigonometry are the Law of Cosines and the Law of Sines. The first of these reads-

$$c^2=a^2+b^2-2ab\cos(\theta)$$

, where a, b, and c are the three sides of any triangle. The simplest way to prove this law is to start with a square of side-length a+b and then place a smaller rotated square into it. This produces-



The four triangles shown have area $4(ab/2)=(a+b)^2-c^2$. So that the Pythagorean Theorem $c^2=a^2+b^2$ holds. Next magnify the bottom right triangle, draw a diagonal line of length b through the triangle and shorten `a` by the amount shown. The multiple application of the Pythagorean Theorem then produces the following picture and equations-



Eliminating g and h from these three equations then yields the Law of Cosines. Note that at theta=pi/2 one recovers the Pythagorean Theorem. An equilateral triangle has cos(theta)=1/2.

Next we look at the Law of Sines. It reads-

$$\sin(A)/a=\sin(B)/b=\sin(C)/c$$

, where A, B, and C are the angles corresponding to sidelengths a, b, c, respectively. Looking at the blue oblique triangle above, we see that sin(B)/b=sin(C)/c =sin(A)/a. Eliminating one of the sides a, b, or c then produces-

$$\sin(A)/a=\sin(B)/b=\sin(C)/c$$

We can also combine the Law of Sines with the Law of Cosines to get-

$$\sin C=c\sin(A)/a \quad \text{and} \quad \cos(C)=(a^2+b^2-c^2)/2ab$$

Squaring both sides and adding yields-

$$\sin(A)=(1/2bc)\mathrm{sqrt}[(2ab)^2-(a^2+b^2-c^2)^2]$$

This allows us to state the value of the sine of a triangle at any vertex.

This last result also allows us to write the area of any triangle in terms of the length of its three sides. To do this we start with the well known vector result that a triangle area T=bc sin(A)/2. So we find-

$$T=(1/4)\mathrm{sqrt}[(2ab)^2-(a^2+b^2-c^2)^2]$$

For the case of a right triangle the second term in the radical will always vanish since it is then just the Pythagorean Theorem. Also one can convert this last result into the Heron Formula-

$$T=\mathrm{sqrt}[s(s-a)(s-b)(s-c)]$$

, with s=(a+b+c)/2 being the half perimeter. I remember back in high school some 68 years ago asking my math teacher how Heron's Formula was derived. She had no idea.

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