APPLICATION OF THE LEIBNIZ RULE FOR INTEGRATION

In your first course in integral calculus you were introduced to certain integrals of the form-

$$I(x)=\int_{u(x)}^{v(x)} f(x,t)dt$$

which at first glance appeared to be not solvable by standard means such as integration by parts. However by taking a partial derivative with respect to the variable x one often obtains a form which can be directly integrated to find I(x).This differentiation method was first discovered by the German polymath Gottfried Whilhelm Leibniz(1646-1716) three hundred years ago and used extensively by millions, including the physicist and Nobel prize winner Richard Feynman (1918-1988). I myself have a weak connection to G.W.Leibnitz by having been baptized in the same St.Nicholas Lutheran church in Leipzig. Leibniz was the co-inventor of calculus, developed the binary number system, introduced the integral sign, and was the first to show that a two variable function f(x,t) satisfies the Leibniz Rule-

$$\frac{I(x)}{dx} = \int_{u(x)}^{v(x)} \frac{\partial(f(x,t)}{\partial x} dt + \frac{dv(x)}{dx} f\big(x, v(x)\big) - \frac{du(x)}{dx} \mathrm{f(x,u(x))}$$

This rule is easy to establish by simply differentiating I(x) with respect to x. It is the purpose of this note to demonstrate the power of the method by going through some specific examples.

Let us begin by looking at –

$$I=\int_0^\infty \frac{\sin(t)}{t} dt$$

Here we first create the two variable function sin(t)exp(-xt)/t   and then take the derivative with respect to x. This yields-

$$\frac{dI}{dx} = - \int_0^\infty \sin(t) \exp(-xt)\, dt$$

We recognize that this last integral represents the Laplace transform of sin(t) with s=x. So we get-

$$\frac{dI}{dx}=\frac{-1}{1+x^2}$$

On integrating we have-

$$I=\arctan(\infty)-\arctan(0)=\frac{\pi}{2}$$

A second integral we can handle by integration under the integral sign (now often referred to as the Leibniz-Feynman Mehod) is the integral-

$$I(x)=\int_{t=0}^{\infty}\exp(-xt)\,/sqrt(t)dt$$

Differentiating once under the integral with respect to x we find-

$$dI(x)/dx=-\int_{t=0}^{\infty}sqrt(t)\exp(-xt)\,dt$$

Recognizing that this last integral is just the Laplace transform for sqrt(t) when s=x, we find

$$\frac{dI}{dx}=-\frac{\Gamma(\frac{3}{2})}{x\hat{}(\frac{3}{2})}=-\frac{sqrt(\pi)}{2x\hat{}(\frac{3}{2})}$$

On integrating, one obtains the closed form result-

$$I(x)=sqrt(\pi/x)$$

Finally we look at a case where one of the integral limits is a function of x. The case we have in mind is-

$$I(x)=\int_{t=0}^{t=x}sqrt(x+t)dt$$

Carrying out a differentiation with respect to x we get-

$$\frac{d(I)}{dx}=\int_{t=0}^{x}\frac{d}{dx}(sqrt(x+t)dt+sqrt(2x)$$

On integrating we obtain the exact analytic solution-

I(x)=(2/3)(2sqrt(2)-1)x^(3/2)

To check this answer at x=1, we integrate the integral directly to get the same answer-

$\int_{t=0}^{1} sqrt(1+t)$dt=(2/3)(2sqrt(2)-1)=1.21895…

Note that for this last case we could also have solved the integral I(x) directly by the substation z=x+t without needing to differentiate under the integral.

There are an infinite number of other forms for f(x,t) with specified u(x) and v(x). The three examples I have given above should suffice to demonstrate the power of the method.

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ps-The correct pronunciation of Leibniz has the ei sound as in the German word Leib(body). On the U Tube you will often hear the wrong pronunciation using the reversed ie as in Liebe(love).

My own last name Kurzweg is constructed from the German words kurz(short) and Weg(path) and is correctly pronounced as the two words hooked together. I remember as an undergraduate at the University of Maryland that my math teacher Monroe Martin used to refer to me as Mr Geodesic. Very clever, since the shortest distance between two points on a globe is known as a geodesic.