LIMITS OF QUOTIENTS USING THE L'HOSPITAL RULE

Most of you learned back during your introductory course in calculus that the limit of sin(x)/x=1 at x=0. You probably got this result by writing out sin(x)=x-x^3/3!+x^5/5!- and then dividing by x and finally evaluating at x=0. This procedure represents the well known L'Hospital Rule which says that the limit of a quotient f(x)/g(x) when its value equals 0/0 or infinity/infinity is given by-

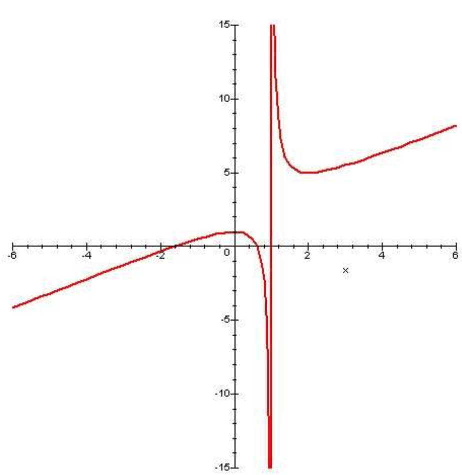
$$\text{limit } x \rightarrow x_0 \ [f'(x_0)/g'(x_0)] \text{ or } [f''(x_0)/ g''(x_0)] \text{ etc}$$

One stops with the lowest derivative quotient which does not equal 0/0 or infinity/infinity. Guilaume l'Hospital (1661-1704) was a French aristocrat and mathematician who wrote the first French calculus book. Some claim that it was actually J.Bernoulli who came up with the idea of the LHospital Rule and that L'Hospital paid him for it.

We want in this note to demonstrate the L'Hospital Rule for several different examples.

Let us begin with lim x→1 [(x^3-2*x+1)/(x-1)^2] . Here we have 0/0 at x=1. So the derivative quotient becomes [(3x^2-2)/2(x-1)]=1/0=infinity. So the function has an infinity at x=1. Here is its graph-

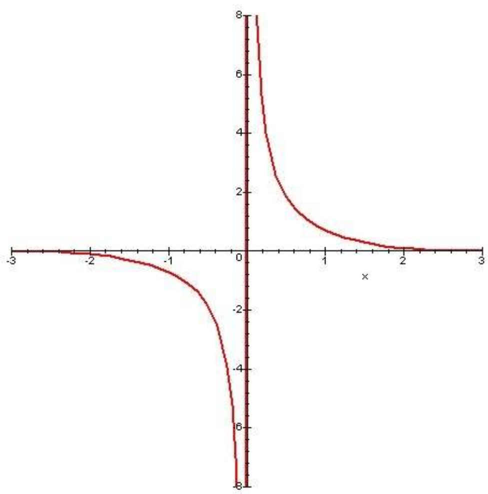
Plot of (x^3-2x+1)/(x-1)^2 SHOWING INFINITY AT x=1



Consider next the function $F(x)=(\exp(x)-1)/x$. At $x=0$ we find $F(0)=\lim\rightarrow(0/0)$. Hence $F(0)=\exp(0)/1=1$.

Consider next a case where both $f(x)$ and $g(x)$ plus $f'(x)$ and $g'(x)$ vanish at $x=0$. The function $F(x)=\sin(x)^2/x^3$ will do this to yield $F(0)=$infinity based upon $f(0)''$ and $f(0)''$. Here is the graph of $F(x)$-

FUNCTION F(x)=sin(x)^2/x^3



here limit F(0)= f(0)"/g(0)"= infinity

Finally we examine the function-

$$F(n)=[1+(1/n)]^n=(n+1)^{(n-1)}/n^{(n-1)}$$

and ask for its value in the limit of n equal to infinity. It is seen that all derivative terms involving f(infinity) plus g(infinity) vanish. So L'Hospital's Rule seems not to work directly. However one can carry out a binomial expansion of F(infinity)to get-

$$F(\text{infinity})=1+1+1/2!+1/3!+1/4!+\ldots=\sum_{k=0}^{\infty}\frac{1}{k!}=\exp(1)$$

This leaves one with the well known identity-

$$\lim_{n\to\infty}(1+1/n)^n=\exp(1)$$

To see how the n function approaches exp(1), we have constructed the following table-

n	(1+1/n)^n
1	2
10	2.593
100	2.70481
1000	2.7169239
infinity	2.718281828459045235

The convergence rate is seen to be rather slow.

Those of you familiar with compound interest will recognize that the n function given above as n goes to infinity is just the return on one's capital when things are compounded on a continual basis. I recall the value of exp(1) out to 32 places using an event mnemonic constructed by us several years ago. It reads-

2.7-AndrewJackson twice-right triangle –fibonacci three-full circle-year before crash-Boing jet-end of black death -road west

2. 7   18281828   459045   235   360

28   747   1352   66

e= 2.71828182845904523536028747135266

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