PROPERTIES OF A NEW TYPE OF PASCAL TRIANGLE

It is well known that the standard Pascal Triangle contains elements in its nth row equivalent to the binomial coefficients of the polynomial f(x)=(1+x)^n. In playing around with this relation about a decade ago, it became clear to us that one should be able to construct an infinite number of other modified Pascal triangles by just looking at the Taylor series expansion of other f(x)s. Among these we found a new, and heretofore unknown, modified Pascal Triangle based on the function f(x)=1/(1-exp(x)). (see https://mae.ufl.edu/~uhk/MORE-PASCAL.pdf). It led to the following infinite row triangle-

MODIFIED PASCAL TRIANGLE

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
1 57 302 302 57 1
1 120 1191 2416 1191 120 1
1 247 4293 15619 15619 4293 247 1
1 502 14608 88234 156190 88234 14608 502 1

D[n,m]=(n+1-m)D[n-1,m-1]+mD[n-1,m]

It is the purpose of this note to discuss the properties of this triangular structure and then use it to derive some new identities.

We begin by noting that the elements are symmetric about a vertical axis passing through 1, 4, 66, 2416, 156190, etc. In the diagram we show the first nine rows n=1 through 9. One notes that for a given n the integer values of m range from 1 through n. In playing around with the elements D[n,m] of this triangle one can express-

D[,n m]=(n+1-m)D[n-1,m-1]+mD[n-1,m]

Except for the constants multiplying the Ds, this identity looks a lot like that for the standard Pascal Triangle where D[n,m]=D[n-1,m-1]+D[n-1,m]. To demonstrate, Let us find D[7,4].We get-

D[7,4]=(7+1-4)D[6,3]+4D[6,4]=4(302)+4(302)=2416

Next we add up the integers in a given row. This produces 1, 2, 6, 24, 120, 720, 5040, 40320, 362880 .These are recognized as the factorials of n one through nine. So the sum of all elements in row n equal to n!. We will make use of this fact later in this article.

It is also possible to calculate D[n,m] directly by evaluating a single finite series extending from k=1 through m. It reads-

$$D[n,m]=\sum_{k=1}^{m}\{(-1)^{k-1}(n+1)!\,(m+1-k)\text{^}n\}/\{(k-1)!\,(n+2-k)!\}$$

Evaluating this sum for n=9 and m=3 produces D[9,3]=14608 in agreement with the above graph. Also we have D[11,6]=15724248 and D[15,8]=447538817472. With aide of a math program such as MAPLE these results are obtainable in split seconds. You will also note that those rows with 2^n elements have each element equal to an odd number. Here is the result for n=16 and $1\le m \le 16$ –

SIXTEEN ODD ELEMENTS IN ROW n=16

$$\text{D[n,m]}=\sum_{k=1}^{m}\frac{(-1)^{(k-1)}(n+1)!\,(m+1-k)^n}{(k-1)!\,(n+2-k)!}$$

m	D[16,m]
1	1
2	65519}
3	41932745}
4	3572085255
5	85383238549
6	782115518299
7	3207483178157
8	6382798925475
9	6382798925475
10	3207483178157
11	782115518299
12	85383238549
13	3572085255
14	41932745
15	65519}
16	1

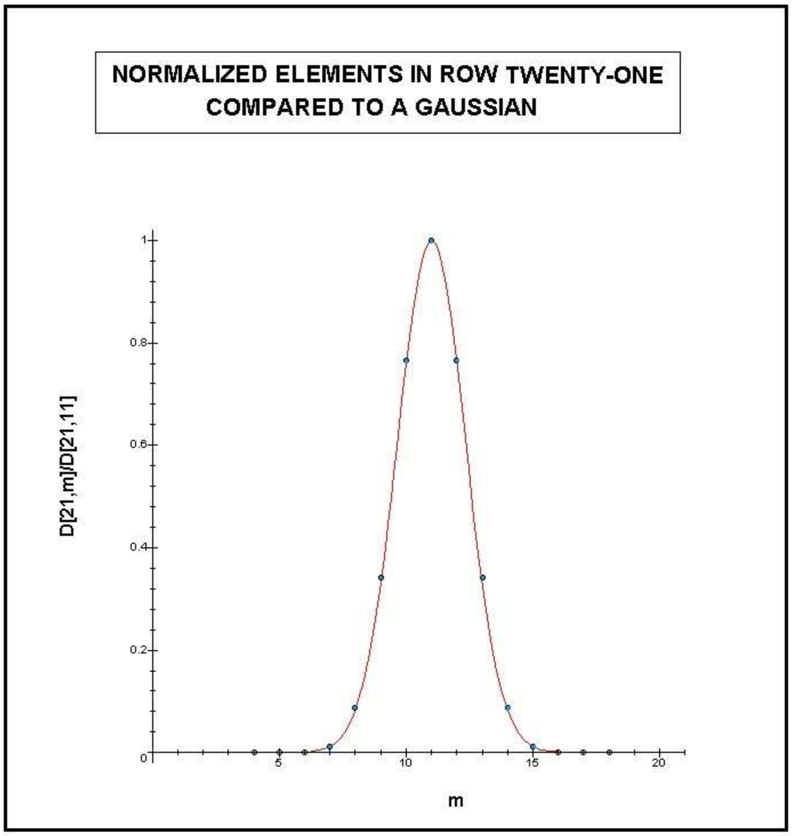
Finally we come to the most interesting feature of the modified Pascal Triangle. When looking at any row, as we have already done above, the sum of the elements equal to n!.

Also the elements in a given row reach a maximum value at m=n/2 and drop- off rapidly in value in a symmetric manner toward one at both m=1 and m=n. This feature suggests that a reasonable approximation to n! can be gotten by simply adding together a few terms at m=n/2 and near these such as at m=n/2$\pm$1 , $m=\frac{n}{2}\pm 2, and\ m=\frac{n}{2}\pm 3$. For the special case of n=11 we get-

$$11!\approx .15724248*10^{\wedge}8+2*(.9738114*10^{\wedge}7+.2203488*10^{\wedge}7+152637)= 39912726$$

This lies close to the exact value n!= 39916800.

It is also possible to normalize D[n,m] and plot the result for fixed say n=21. Here is the result-



The blue circles represent the normalized D[21,m] elements. Upon these we have superimposed the Gaussian-

G:=exp(-0.2764345269(m-11)^2)

, where the constant has been adjusted to pass exactly through the two points [n,m]= [21,11] and [21,7]. The agreement between location of the blue point circles and the continuous Gaussian is amazing.

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February 29, 2024
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