SIGMA AND RELATED FUNCTIONS

The sigma function , also referred to as the divisor formula, equals the sum of all divisors of an integer N. Thus-

$$\sigma(2^4)=\sigma(16)=1+2+4+8+16=31=2^5-1$$

and-

$$\sigma(3^4)=\sigma(81)=1+3+9+27+81=121$$

Recallng the definition of one of our earlier created functions f(N)=[sum(all divisors) –(N+1)]/N, we can also relate the sigma function to the number fraction f(N) as-

$$\sigma(N)=1+N+Nf(N)$$

Thus f(16)=[31-1-16]/16=7/8.

If N equals a prime p, we have-

$$\sigma(p)=1+p$$

$$\sigma(p^2)=1+p+p^2$$

$$\sigma(p^3)=1+p+p^2+p^3$$

and

$$\sigma(p^n)=\sum_{k=0}^{n} p^k$$

Also if N =pq is a semi-prime with p < q being primes, we find-

$$\sigma(N)=1+p+q+N$$

The corresponding number fraction has the definition value-

$$f(N)=(p+q)/N=(p+q)/(pq)$$

and thus lies close to zero when N gets large. A number of importance in the factoring of large semi-primes is-

$$S=(p+q)/2=(1/2)[\sigma(N)-N-1]$$

One finds the factored primes to be-

$$[p,q]=S\mp\sqrt{S^2-N}$$

So, if N=77 we get S=9 and [p,q]=[7,11].

Note that for semi-primes σ(N) will always be known if p amd q are known. For example, p=3217 and q=6291 produces-

$$\sigma(N)=1+p+q+pq=21846647$$

Most advanced mathematics programs contain the value of sigma(N) for Ns up to about forty digit length. This means all semi-primes smaller than this size will have S known and hence the factors [p,q] determinable in a split second.

To quickly find the sigma function of any number, one recalls that any integer can always be written as the product of powers of component primes. Thus the number –

$$N=314836=2^2x31x2539$$

will have σ(N) given as-

$$\sigma(N)=\sigma(2^2)\sigma(31)\sigma(2539)=7x32x2540=568960$$

Another interesting result involving the sigma function occurs for σ[2^(p-1)]. Here we have upon expansion-

$$\sigma[2^{(p-1)}]=\{1+2+2^2+\ldots+2^{(p-1)}\}=2^p-1$$

The term on the right of this equality is just a Mersenne Number M(p) which will be a prime only for certain restricted values of p. Two possible Mersenne Primes are-

$$M(5)=2^5-1=31 \quad \text{and} \quad M(7)=2^7-1=127$$

Note that all Mersenne Numbers must have the form 6n+1 so that M(p) mod(6)=1 regardless if M(p) is prime or not. For example, the non-prime M(9)=511=23x89 still has M(9) mod(6)=1.

We also find from the formulas above that –

$$2^n-1=\sum_{k=0}^{n-1} 2^k+=1+2+4+\ldots+2^{(n-1)}$$

Thus-

$$1+2+4+8+16+32+64+128+256=2^8-1=511$$

If we replace 2 by 5 in this last expression we find-

$$\sigma(5^n)=1+5+25+125+\ldots+5^n$$

For n=3, this yields sigma(125)=156 which is not a recognizable power of 5 minus one.

Finally consider the difference of two sigma functions based on powers of the primes q and p. We have-

$$\sigma(q^n)-\sigma(p^n)=(q-p)+(q^2-p^2)+(q^3-p^3)+\ldots+(q^n-p^2)$$

Thus for q=5 and p=3 with n=3, we have the identity-

$\sigma(125)-\sigma(27)=2+16+68=116$

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