SOLVING QUADRATIC AND CUBIC EQUATIONS USING THEIR DEPRESSED FORMS

Two non-linear algebraic equations which one is first exposed to in early high school is the quadratic equation-

$$ax^2+bx+c=o$$

and the cubic equation-

$$ax^3+bx^2+cx+d=0$$

The first of these can easily be solved by completing the square on x. The solution reads-

$$x= [-b \pm \sqrt{b^2-4ac}]/2a$$

It is a result committed to memory by most high school students. The constants appearing in the solution are given beforehand and may be complex. The cubic equation has a much more complicated form and never derived in class but referred to in handbooks and evaluated via pocket calculators. We want in this note to give the full analytic solution to both of these two non-linear equations using their depressed form not containing the power of $x^{(n-1)}$.

QUADRATIC EQUATION:

Here we start with the substitution x=t+k to get-

$$t^2+t[2k+(b/a)]+[(ak^2+bk+c)]/a=0$$

On setting k=-b/(2a) , we get the depressed quadratic-

$$t^2=[\{b^2-4ac)/(4a^2)]$$

Taking the root we get the well known result-

$$x=t-\frac{b}{2a}=\left[-b \pm \sqrt{b^2-4ac}\right]/(2a)$$

Note that there will be two real roots if a, b, and c are real and $b^2>4ac$. A negative radical means two complex roots.

CUBIC EQUATION:

On letting x=t+k, we find-

$$at^3+t^2[3ak+b]+t[3ak^2+2bk+c]+[ak^3+bk^2+ck+d]=0$$

Next letting k=-b/(3a), the t^2 term vanishes and we get-

$$t^3+mt=n$$

with the known quantities-

$$m=[(3ac-b^2)/(3a^2)] \quad \text{and} \quad n=[(-27da^2+9abc-2b^3)/(27a^3)]$$

The renaissance mathematicians N.Tartaglia(1500-1557) and G.Cardano(1501-1576), where the first to discover this depressed form of a cubic. To solve this simplified form one introduces the new variables-

$$u+v=t \quad \text{and sets} \quad 3uv+m=0$$

This produces - $u^3+v^3=n$ with $v=-m/(3u)$

Getting rid of the v term we end up with the effective quadratic equation in z given as-

$$z^2-nz-m^3/27=0 \quad \text{where } z=u^3$$

On solving , we find –

$$z=u^3=(1/2)[n \pm \text{sqrt}(n^2+4m^3/27)]$$

Hence our original cubic equation has the solutions-

$x=u-m/(3u)-b/(3a)$ with u being the three root of z given above.

SOLUTIONS OF A CUBIC:

To demonstrate the validity of the above solution consider the cubic-

$$x^3-2x^2-x+2=0$$

Here $a=1$, $b=-2$ $c=-1$, and $d=2$. This produces the depressed cubic equation

$$t^3-7t/3=-20/27$$

,where $m=-7/3$ and $n=-20/27$. So we have as a solution-

$t=u+7/(9u)$ with $u^3=[-20\pm\text{sqrt}(-972)]/54$

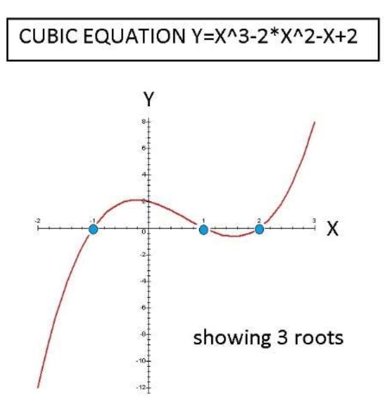
This produces-

$$t=1.6666667 \quad \text{and} \quad x=1.000000$$

So one of the three rootes is $x_1=1$ The other two follow from the quadratic-

$$(x^3-2*x^2-x+2)/(x-1)= x^2-x-2=0$$

This gives the remaining roots as $x_2=2$ and $x_3=-1$. Sometimes the remaining two roots are complex conjugates. This was something the mathematicians of the 15 hundreds were not aware of. Here is a graph of $y=x^3-2x^2-x+2$ showing the three real roots of this equation-



If the sketched cubic has only one crossing of y=0 then the other two are complex conjugates.

There were efforts later to find analytic solutions to the general non-linear 4^{th} and 5th order algebraic equations. The fourth was found using a similar reduction approach used here. But no analytic solutions for 5^{th} and higher power were ever discovered. It was the briliand Norwegian mathematician N. Abel(1802-1829) who first showed that it is impossible to find closed form solutions for the general 5^{th} and higher order polynomial algebraic equations. It's too bad Abel died so early of TB (the Covid 19 of his day). A lot more could have been accomplished by him.

Today of course finding the roots to any order algebraic equation is an easy matter to carry out via calculus, electronic computer and/or pocket calculator. The procedure there involves simple iteration.

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